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THE CATEGORY OF A MAP AND THE GRADE OF A MODULE*

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ABSTRACT

Let $f : Y \to X$ be a continuous map between connected CW complexes. The homology $H_*(F)$ of the homotopy fibre is then a module over the loop space homology $H_*(\Omega X)$.

THEOREM: *If $H_*(F;R)$ and $H_*(\Omega X;R)$ are R-free (R a principal ideal domain) then for some $H_*(\Omega X;R)$-projective module $P=P_{\geq 0}$ and for some $m \leq \operatorname{cat} f$:*

$$\operatorname{Ext}^m_{H_*(\Omega X)}(H_*(F);P) \neq 0.$$

Some applications are also given.

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1. Introduction

If

$$f\colon Y \to X$$

is a continuous map between connected CW complexes, then the natural right action of the loop space ΩX on the homotopy fibre F of f makes $H_*(F)$ into a right module over the graded algebra $H_*(\Omega X)$. A second important homotopy invariant of f is its **Lusternik–Schnirelmann category**, cat f: the least $m \leq \infty$ such that Y can be covered by $m+1$ open sets U_i with each $f|_{U_i}$ homotopically constant. The main purpose of this paper is to establish a relation between cat f and certain homological invariants of the module $H_*(F)$, namely its **projective dimension** and its **positive projective grade**.

These invariants are defined as follows. Here and throughout we work over a fixed principal ideal domain R (so that $H_*(-;R), \otimes_R$ and Hom_R are denoted respectively by $H_*(-)$, $\otimes$ and Hom). Modules over graded algebras A are by definition graded and if M, N are A-modules then

$$\mathrm{Hom}_A(M;N) = \{\mathrm{Hom}_A(M;N)^i\}_{i\in\mathbb{Z}}$$

is the graded R-module whose i^{th} component $\mathrm{Hom}_A(M;N)^i$ consists of the A-linear maps sending each M_j to N_{j-i}. The corresponding derived functors $\mathrm{Ext}_A^q = \{\mathrm{Ext}_A^{q,i}\}_{i\in\mathbb{Z}}$ also take values in graded R-modules; i is called the **internal degree.**

The **projective dimension** of an A-module M is the greatest n (or ∞) such that $\mathrm{Ext}_A^n(M;-) \neq 0$. The **projective grade** of M is the least m (or ∞) such that $\mathrm{Ext}_A^m(M;V) \neq 0$ for some projective A-module, V. The **positive** (resp. **negative) projective grade** of M is the least m (or ∞) such that $\mathrm{Ext}_A^m(M;V) \neq 0$ for some A-projective V generated by $V_{\geq 0}$ (resp. by $V_{\leq 0}$). Thus $\mathrm{proj.grade}_A(M)$ is the lesser of the positive and negative projective grades.

Observe also that if $A = \{A_i\}_{i\geq 0}$ and if $M = \{M_j\}_{j\geq -q}$ then

$$\mathrm{pos.proj.grade}_A(M) \leq \mathrm{proj.dim}_A(M). \tag{1.1}$$

Indeed, neither side of (1.1) is affected if degrees in M are increased by q; i.e. we may assume $M = \{M_j\}_{j\geq 0}$. Then M has a projective resolution of the form $\cdots \to P_k \xrightarrow{d} P_{k-1} \to \cdots$ with each P_k concentrated in non-negative degrees. Now if $\mathrm{proj.dim}_A(M) = n$ then $K_n = \ker(d\colon P_{n-1} \to P_{n-2})$ is A-projective, so that $0 \to K_n \to P_{n-1} \to \cdots$ is also a projective resolution. Moreover the identity map of K_n represents a non-zero class in $\mathrm{Ext}^n_A(M; K_n)$, which establishes (1.1).

We can now state our main result:

THEOREM A: *Let F be the homotopy fibre of a continuous map $f\colon Y \to X$ between path connected CW complexes, and suppose $H_*(F)$ and $H_*(\Omega X)$ are both R-free. Then*

(i) $\mathrm{pos.proj.grade}_{H_*(\Omega X)}(H_*(F)) \le \mathrm{cat}\, f$.

(ii) *When equality holds in (i) then also*

$$\mathrm{pos.proj.grade}_{H_*(\Omega X)}(H_*(F)) = \mathrm{proj.dim}_{H_*(\Omega X)}(H_*(F)).$$

To compare this with earlier results we need first to recall for modules M over graded algebras A that $\mathrm{grade}_A(M)$ is the least m (or ∞) such that $\mathrm{Ext}^m_A(M; A) \neq 0$. And the depth (resp. (pos.)proj.depth, cohomological dimension) of an augmented graded algebra $A \to R$ is the grade (resp. (pos.) proj. grade, proj. dimension) of the trivial A-module, R.

Evidently $\mathrm{pos.proj.grade}_A(M) \le \mathrm{grade}_A(M)$. Moreover, equality holds if R is a field, $A = \{A_i\}_{i \ge 0}$ and each A_i is finite dimensional over R:

$$\mathrm{pos.proj.grade}_A(M) = \mathrm{grade}_A(M). \tag{1.2}$$

In fact if $\mathrm{pos.proj.grade}_A(M) = m$ then $\mathrm{Ext}^m_A(M; P) \neq 0$ for some non-negatively generated A-projective P. Now P is a retract of a free A-module $V \otimes A$ with $V = V_{\ge 0}$. And if v_α is a basis of V then V is a retract of the vector space $\hat{V} = \prod_\alpha R \cdot v_\alpha$, where we have taken the product in the category of graded vector spaces. It follows that $\mathrm{Ext}^m_A(M; \hat{V} \otimes A) \neq 0$. Finally, our hypothesis on A implies that $\hat{V} \otimes A = \prod_\alpha A \cdot v_\alpha$. Hence $\mathrm{Ext}^m_A(M; \hat{V} \otimes A) = \prod_\alpha \mathrm{Ext}^m_A(M; A) \cdot v_\alpha$, and (1.2) follows.

Now in Theorem A consider the special case that R is a field and f is the identity map of a simply connected CW complex X for which each $H_i(X; R)$ is finite dimensional. The conclusions then read: $\mathrm{depth}\, H_*(\Omega X) \le \mathrm{cat}\, f$, with equality implying $\mathrm{depth}\, H_*(\Omega X) = \mathrm{coh.dim}\, H_*(\Omega X)$. This is precisely Theorem A of [6], established earlier in [5] for the case $R = \mathbb{Q}$. Thus our present result extends the earlier one by:

(i) Passing from spaces to maps.

(ii) Extending to the non-simply connected case.

(iii) Removing the hypotheses that $H_*(X)$ be of finite type and that R be a field.

The hypothesis of finite depth leads to general structure theorems for loop space homology, as is shown in [6], [7], [8] and [9]. Thus the restricted form of Theorem A in [6] implies that when X is a 1-connected CW complex of finite type and finite LS category then $H_*(\Omega X)$ satisfies these structure theorems. With the aid of Theorem A we can now give (Theorem B below) an additional large class of spaces where these theorems hold. And, as an illustration in the non-simply connected case, we also give an application (Theorem C) to the properties of $\pi_1(X)$ when X and its universal cover have the homotopy type of finite complexes. Other, less immediate, applications will be established in a subsequent paper.

THEOREM B: *Let $f\colon Y \to X$ be a continuous map between path connected CW complexes, such that $H_*(\Omega f)$ is R-split injective. If $H_*(\Omega X)$ is R-free then*

$$\text{pos.proj.depth}\, H_*(\Omega Y) \leq \text{cat}\, f.$$

If equality holds then also $\text{pos.proj.depth}\, H_*(\Omega Y) = \text{coh.dim}\, H_*(\Omega Y)$.

Proof: Let F be the homotopy fibre of f. There is then an ΩY-principal fibration $\Omega Y \xrightarrow{\Omega f} \Omega X \xrightarrow{\rho} F$, with ρ an ΩX-equivariant map. When $H_*(\Omega X)$ is R-free a standard Serre spectral sequence argument shows that the following conditions are equivalent:

(i) $H_*(\Omega f)$ is R-split injective,

(ii) The Serre spectral sequence for ρ collapses at $E^2_{*,*} = H_*(\Omega Y) \otimes H_*(F)$ and $E^2_{*,*}$ is R-free;

(iii) $H_*(\rho)$ is R-split surjective and $\pi_1(\rho)$ is surjective. (Note that this holds even in the slightly complicated case when X and Y are not simply connected, so that F may not even be path connected.)

Thus under the hypotheses of the theorem, $H_*(F)$ is R-free and $H_*(\rho)$ factors to give an isomorphism,

$$R \otimes_{H_*(\Omega Y)} H_*(\Omega X) \xrightarrow{\cong} H_*(F),$$

of right $H_*(\Omega X)$-modules. For the sake of simplicity, denote $H_*(\Omega Y) \subset H_*(\Omega X)$ by $K \subset G$. Then

$$\text{(1.3)} \qquad \text{Ext}_G(H_*(F); -) = \text{Ext}_G(R \otimes_K G; -) = \text{Ext}_K(R; -)$$

since G is left K-free.

But reversing loops in Y and X induces anti-isomorphisms of the homology algebras K and G, so that G is also right K-free. Moreover any K-projective generated in degrees ≥ 0 is a retract of some $V \otimes K$, $V = V_{\geq 0}$ and hence of $V \otimes G$. Thus (1.3) gives

$$\text{(1.4)} \qquad \text{pos.proj grade}_{H_*(\Omega X)}(H_*(F)) = \text{pos.proj.depth}(H_*(\Omega Y)),$$

and so the first assertion of Theorem B reduces to Theorem A(i).

For the second assertion let P_* be a G-free resolution of R, and consider the exact sequence

$$o \to \ker d_n \to P_n \to \operatorname{Im} d_n \to 0$$

of G-modules. If $\operatorname{cat} f = \text{pos.proj.depth}(K) = \text{pos.proj.grade}_G(H_*(F)) = n$, it follows from Theorem A(ii) and from (1.3) that $\operatorname{Ext}^1_K(\operatorname{Im} d_n; N) = 0$ for any G-module, N. In particular $\operatorname{Ext}^1_K(\operatorname{Im} d_n; \ker d_n) = 0$ Thus the short exact sequence above splits over K and so $\text{coh.dim}(K) = n$. ∎

Remark 1.5: (i) As observed in the proof, the hypothesis that $H_*(\Omega f)$ be R-split injective is equivalent to: $H_*(F)$ is R-free, $\pi_1(\rho)$ is surjective and $H_*(\Omega X)$ acts transitively on $H_*(F)$.

(ii) Formula (1.4) furnishes many examples of cyclic modules $H_*(F)$ for which $\text{grade}_{H_*(\Omega X)}(H_*(F)) > \operatorname{depth} H_*(\Omega X)$. For instance let f be the inclusion $Y \to Y \vee Z = X$ with Y, Z simply connected spaces of finite type having non-trivial homology with coefficients in a field, R. Then $H_*(\Omega X) = H_*(\Omega Y) \amalg H_*(\Omega Z)$ has depth one. On the other hand $H_*(\Omega Y)$ can have arbitrarily high depth. (If Y is a product of spheres then depth $H_*(\Omega Y)$ is the number of factors.) And formula (1.2) together with (1.4) gives $\operatorname{depth} H_*(\Omega Y) = \text{grade}_{H_*(\Omega X)}(H_*(F))$. ∎

We come next to our application of Theorem A to fundamental groups. Recall [1; Chap. VIII] that an $FP_\infty(R)$ group is a discrete group Γ such that R has a projective $R[\Gamma]$-resolution in which each term is a finitely generated $R[\Gamma]$-module.

THEOREM C: *Suppose the universal cover, $\tilde{X}$, of a connected CW complex X satisfies:* $\dim H_i(\tilde{X}; \mathbb{Z}_p) < \infty$, *all i, for some fixed prime p. Suppose*

$$1 = \Gamma_0 \triangleleft \Gamma_1 \triangleleft \cdots \triangleleft \Gamma_n \subset \pi_1(X)$$

is a sequence of subgroups with each Γ_i normal in Γ_{i+1}. If each Γ_i/Γ_{i-1} is an infinite $FP_\infty(\mathbb{Z}_p)$-group, then

$$\operatorname{cat} X \geq n.$$

COROLLARY: $\dim X \geq n$.

Proof: Pass to a suitable cover to reduce to the case $\Gamma_n = \pi_1(X)$. Then apply Theorem A to the classifying map $f\colon X \to K(\Gamma_n; 1)$, noting that $\operatorname{cat} f \leq \operatorname{cat} X$, $F = \tilde{X}$ and $\Omega K(\Gamma_n, 1) \simeq \Gamma_n$, to conclude that for some i,

$$\text{pos.proj.grade}_{\mathbb{Z}_p[\Gamma_n]}(H_i(\tilde{X}; \mathbb{Z}_p)) \leq \text{cat} X.$$

Let $G \triangleleft \Gamma_n$ be the normal subgroup of elements acting trivially on $H_i(\tilde{X}; \mathbb{Z}_p)$. Since $H_i(\tilde{X}; \mathbb{Z}_p)$ is finite by hypothesis, G (resp. $G_k = G \cap \Gamma_k$) has finite index in Γ_n (resp. in Γ_k). It follows as in [1; VIII, Prop. 5.1] that each G_k/G_{k-1} is an infinite $FP_\infty(\mathbb{Z}_p)$-group. The standard "melding of projective resolutions" argument then shows that each G_k is an $FP_\infty(\mathbb{Z}_p)$-group.

This in turn implies [1; VIII, Theorem 4.8] that $\text{Ext}_{\mathbb{Z}_p[G_k]}(\mathbb{Z}_p; -)$ commutes with direct limits. In particular

$$\text{depth}\, \mathbb{Z}_p[G] = \text{pos.proj.depth} \mathbb{Z}_p[G].$$

On the other hand the Hochschild–Serre spectral sequence shows that

$$\begin{aligned} \text{pos.proj.depth}\, \mathbb{Z}_p[G] &= \text{pos.proj.grade}_{\mathbb{Z}_p[G]}(H_i(\tilde{X}; \mathbb{Z}_p)) \\ &\leq \text{pos.proj.grade}_{\mathbb{Z}_p[\Gamma_n]}(H_i(\tilde{X}; \mathbb{Z}_p)), \end{aligned}$$

and so altogether

$$\text{depth}\, \mathbb{Z}_p[G] \leq \mathit{cat} X.$$

Theorem C will now follow from the inequalities

$$\text{depth}\, \mathbb{Z}_p[G_k] < \text{depth}\, \mathbb{Z}_p[G_{k+1}]. \tag{1.6}$$

To establish these we use the $FP_\infty(\mathbb{Z}_p)$ condition to identify

$$\text{Ext}_{\mathbb{Z}_p[G_k]}(\mathbb{Z}_p; \mathbb{Z}_p[G_{k+1}])$$

as a free $\mathbb{Z}_p[G_{k+1}/G_k]$-module M. Since G_{k+1}/G_k is infinite,

$$\text{Ext}^0_{\mathbb{Z}_p[G_{k+1}/G_k]}(\mathbb{Z}_p; M) = 0.$$

Thus (1.6) follows from the Hochschild–Serre spectral sequence. ∎

Remark: Note that the action of $\pi_1(X)$ on $H_*(\tilde{X})$ that arises in Theorem C is that induced by the covering transformations. ∎

The rest of this paper is taken up with the proof of Theorem A. In §2 we recall basic definitions including certain properties of the classical Eilenberg–Zilber maps and their application to chains on joins and Hopf spaces. In §3 we remark (Theorem 3.1) that when $\operatorname{cat} f = m$ there is an ΩX-equivariant map from the homotopy fibre F to the m^{th} iterated join $(\Omega X)^{*m}$; this is essentially an observation of Ganea's. This observation is then translated into a factorization theorem for $C_*(\Omega X)$-modules (Theorem 4.8) using the machinery of §2. Finally, in §5 we prove Theorem A via some homological calculations starting from Theorem 4.8.

2. Conventions and preliminary observations

We work over a principal ideal domain R and write Hom and $\otimes$ for Hom_R and $\otimes_R$. Graded objects $M = \{M_i\}$ (including chain complexes, graded and differential graded algebras, etc.) are $\mathbb{Z}$-graded and the degree of $m \in M$ is written $|m|$. Differential graded algebras are called DGA's.

Chain complexes have differentials of degree -1 and we also use Hom and $\otimes$ to denote the corresponding functors in the (differential) graded context; e.g. $\mathrm{Hom}(M;N)^{-p} = \mathrm{Hom}(M;N)_p = \prod_i \mathrm{Hom}(M_i;N_{i+p})$ and $df = d\circ f - (-1)^{|f|} f\circ d$. The **suspension** sM is defined by $(sM)_k = M_{k-1}$ and $dsx = -sdx$. The **homology** functor is denoted $H(-)$ and morphisms f are **homology isomorphisms** (resp. **chain equivalences**) if $H(f)$ is an isomorphism (resp. f has a homotopy inverse).

Algebras have an identity η: $R \to A$ and their multiplication μ is associative. A **(right) A-module** over a DGA is a chain complex M with a chain complex morphism $M \otimes A \to M$ making M into a module over the underlying graded algebra. If N is a second A-module then $\mathrm{Hom}_A(M;N) \subset \mathrm{Hom}(M;N)$ is the sub-chain complex of A-linear maps; in particular a morphism is a cycle of degree zero in $\mathrm{Hom}_A(M;N)$ and two morphisms are **A-homotopic** if their difference is a boundary. An A-module is **A-free** if it has the form $C \otimes A$, with C an R-free chain complex and action by right multiplication; if the differential in C is zero it is A-free on a **basis of cycles.** An **augmented A-module** over an augmented DGA: $A \xrightarrow{\varepsilon} R$ is a surjective morphism $M \xrightarrow{\varepsilon} R$ of A-modules. The **augmentation kernel** will be denoted $IM = \ker \varepsilon$. The suspension sM of an A-module is an A-module via $(sx)\cdot a = s(x \cdot a)$.

A **(differential) graded Hopf algebra**, or $(D)GH$, is an augmented

$$DGA\colon A \xrightarrow{\varepsilon} R$$

together with a morphism $\Delta\colon A \to A \otimes A$ (the **diagonal**) satisfying $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$. A **conjugation** is a degree zero chain map $\omega\colon A \to A$ for which $\mu \circ (\omega \otimes \mathrm{id}) \circ \Delta = \mu \circ (\mathrm{id} \otimes \omega) \circ \Delta = \eta\varepsilon$. Conjugations (if they exist) are unique, and anti-isomorphisms of Hopf algebras.

If M and N are A-modules then $M \otimes N$ is an $A \otimes A$-module; hence an A-module via Δ; we say A acts diagonally in $M \otimes N$.

LEMMA 2.1: *Suppose A is a DGH with conjugation ω. If M and N are A-modules and N is A-free then the A-modules $M \otimes N$ (diagonal action) and $M \otimes N$ (action: $(m \otimes n) \cdot a = m \otimes n \cdot a$) are isomorphic.*

In particular if M is R-free then $M \otimes N$ is A-free.

Proof: We may suppose $N = A$ (action by right multiplication). Let $\alpha\colon M \otimes A \to M$ be the action on M. Then the desired isomorphism and its inverse are given by $(\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \omega \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$ and $(\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$. ∎

Fix a DGH, A (possibly $A = R$) and let M be an augmented A-module. The **cone** on M is the augmented A-module $cM = M \oplus sIM$ with $d(x, sy) = (dx + y, -sdy)$. If M' is a second augmented A-module the inclusions ξ, ξ' of M, M' in their cones define an inclusion

$$\hat{\xi} = \xi \otimes \mathrm{id} - \mathrm{id} \otimes \xi'\colon M \otimes M' \to [cM \otimes M'] \oplus [M \otimes cM'],$$

and the **join** $M * M'$ is the augmented A-module

$$M * M' = \frac{[cM \otimes M'] \oplus [M \otimes cM']}{\mathrm{Im}\,\hat{\xi}},$$

with diagonal action.

We now turn to topological spaces where we work entirely in the category of Hausdorff compactly generated spaces X with appropriate product and mapping space topologies as described in [16; I §4]. In particular the Moore path space $MX \subset X^{[0,\infty)} \times [0, \infty)$ consists of those (f, r) with $f(t) = f(r)$ for $t \geq r$, and its properties, with different notation, are described in [16; III, §2]. Thus, writing $- \underset{X}{\times} MX$ and $MX \underset{X}{\times} -$ for the fibre products with respect to $(f, r) \mapsto f(0)$

and $(f,r) \mapsto f(r)$, we note that $PX = MX \times_X \{a\}$ and $\Omega X = \{a\} \times_X MX \times_X \{a\}$ are the path and loop spaces at $a \in X$ and that composition is a continuous map $MX \times_X MX \to MX$ which makes ΩX into a topological monoid acting continuously from the right on PX. Letting $\nu\colon \Omega X \to \Omega X$ be the path reversing map we see that ΩX is a Hopf space in the sense of

Definition 2.2: A **Hopf space** is a topological monoid G with a self map ν such that $x \mapsto x \cdot \nu(x)$ and $x \mapsto \nu(x) \cdot x$ are homotopic to the constant map. An **action** of G on a space X is a continuous map $\alpha\colon X \times G \to X$ which is a right action of the monoid G on the set X. ∎

Hopf space actions of loop spaces arise as follows. Let $\phi\colon Y \to X$ be any continuous map, and fix a base point $a \in X$. Then

$$p\colon E = Y \times_X MX \to X, \qquad p(y,(f,r)) = f(r),$$

is the conversion of ϕ to a fibration and $F = p^{-1}(a)$ is the homotopy fibre of ϕ. Note that $F = Y \times_X PX$ and so composition defines an action of ΩX on F.

We next recall how singular chains $C_*(-) = C_*(-;R)$ convert Hopf spaces and Hopf space actions into differential graded Hopf algebras and modules. First note that the constant map $X \to pt$ provides a natural augmentation $\varepsilon\colon C_*(X) \to R$, while any $x \in X$ determines $j_x\colon R \to C_*(X)$; $j_x(r) = r \cdot x$. We write $IC_*(X) = \ker \varepsilon$. For the multiplication and diagonal we shall need the explicit natural classical Eilenberg–Zilber chain equivalences of augmented chain complexes

$$C_*(X) \otimes C_*(Y) \xrightarrow{\kappa} C_*(X \times Y) \xrightarrow{\lambda} C_*(X) \otimes C_*(Y) \tag{2.3}$$

as defined for instance in [2] and [14; Chap. VIII, Theorems 8.8 and 8.5]: κ is obtained from the standard triangulation on the product of two Euclidean simplexes and λ is the front face-back face map of Alexander and Whitney.

It is standard and straightforward (cf. [14; Chap. VIII, §8]) that κ and λ are strictly associative homotopy inverses and that the Alexander–Whitney diagonal $\Delta\colon C_*(X) \to C_*(X) \otimes C_*(X)$ is just $\lambda \circ C_*(\Delta_X)$, $\Delta_X\colon X \to X \times X$ denoting the topological diagonal. Moreover if

$$\text{inter}\colon M \otimes N \to N \otimes M \text{ and inter}\colon X \times Y \to Y \times X \tag{2.4}$$

are defined respectively by $m \otimes n \mapsto (-1)^{|m||n|} n \otimes m$ and $(x,y) \mapsto (y,x)$ then $\kappa \circ \text{inter} = C_*(\text{inter}) \circ \kappa$ — cf. [14; Chap. VIII, §8]. Less well known, but equally straightforward is the next lemma:

LEMMA 2.5 ([3; §17]): *The following diagram commutes:*

$$\begin{array}{ccc} C_*(X\times X')\otimes C_*(Y\times Y') & \xrightarrow{\text{inter}\circ(\lambda\otimes\lambda)} & C_*(X)\otimes C_*(Y)\otimes C_*(X')\otimes C_*(Y') \\ \Big\downarrow{\scriptstyle\kappa} & & \Big\downarrow{\scriptstyle\kappa\otimes\kappa} \\ C_*(X\times X'\times Y\times Y') & \xrightarrow[\lambda\circ C_*(\text{inter})]{} & C_*(X\times Y)\otimes C_*(X'\times Y') \end{array}$$

Remark 2.6: Regard $C_*(X)$ as a differential graded coalgebra with diagonal Δ. Then Lemma 2.5 implies that κ: $C_*(X)\otimes C_*(Y)\to C_*(X\times Y)$ is a coalgebra morphism. ∎

Now suppose a: $F\times G\to F$ is the action of a Hopf space with multiplication m and identity e. Define

$$\mu = C_*(m)\circ\kappa\colon C_*(G)\otimes C_*(G)\to C_*(G),$$
$$\alpha = C_*(a)\circ\kappa\colon C_*(F)\otimes C_*(G)\to C_*(F).$$

PROPOSITION 2.7: *With the hypotheses and notation above:*

(i) $(C_*(G),\mu,\Delta,\varepsilon,j_e)$ *is a differential graded Hopf algebra.*

(ii) α *makes* $C_*(F)$ *into a* $C_*(G)$*-module.*

Proof: This is an essentially immediate consequence of the observations above about κ and λ, including Remark 2.6. ∎

COROLLARY 2.8:

(i) $H(G)$ *is a graded algebra and* $H(F)$ *is an* $H(G)$*-module.*

(ii) *If* $H(G)$ *is* R*-torsion free then it is a graded Hopf algebra with conjugation.*

Proof: It suffices to use the Kunneth homomorphism $H(C)\otimes H(D)\to H(C\otimes D)$, which is an isomorphism if C,D are R-free and $H(C)$ is R-torsion free. The conjugation is $H(\nu)$, ν the homotopy inverse of G. ∎

If a Hopf space G acts on X it acts on the **cone** $CX = I\times X/\{o\}\times X$ via $(t,x)\cdot g=(t,x\cdot g)$. If G also acts on Y it acts (diagonally) on the product $X\times Y$ and on the **join** $X*Y=(CX\times Y)\cup_{X\times Y}(X\times CY)$. On the other hand, if M and N are any two augmented $C_*(G)$-modules then $M\otimes N$ is a $C_*(G)$-module (diagonal action). Thus the cone cM and join $M*N$ defined above also inherit a natural $C_*(G)$-module structure.

PROPOSITION 2.9: *Suppose a Hopf space G acts on spaces X and Y. Then*

(i) λ: $C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ *and* Δ: $C_*(X) \to C_*(X) \otimes C_*(X)$ *are* $C_*(G)$*-linear.*

(ii) *There is a natural sequence of* $C_*(G)$*-linear chain equivalences of augmented modules, connecting* $C_*(X) * C_*(Y)$ *with* $C_*(X * Y)$.

Proof: The first assertion follows at once from Lemma 2.5. There are natural chain equivalences,

$$cC_*(X) \xrightarrow[\phi]{\simeq} \frac{C_*(I) \otimes C_*(X)}{C_*\{o\} \otimes IC_*(X)} \xrightarrow[\psi]{\simeq} \frac{C_*(CX)}{C_+(pt)} \xleftarrow{\simeq} C_*(CX) \tag{2.10}$$

with ϕ defined by ϕ: $(x, sy) \mapsto \{1\} \otimes x + [0,1] \otimes y$ and ψ induced from λ. Thus ψ is $C_*(G)$-linear by (i) and ϕ is by definition.

On the other hand the inclusions $CX \times Y, X \times CY \to X * Y$ define a $C_*(G)$-linear chain equivalence

$$\frac{C_*(CX \times Y) \oplus C_*(X \times CY)}{C_*(X \times Y)} \xrightarrow{\simeq} C_*(X * Y).$$

We apply λ to the chain complex on the left to replace "$\times$" by $\otimes$, then use (2.10) to replace $C_*(C-)$ by $cC_*(-)$, arriving in this way exactly at $C_*(CX) * C_*(CY)$. ∎

3. Ganea's fibrations

Recall that the join $X * Y$ of two topological spaces is defined by $X * Y = (CX \times Y) \cup_{X \times Y} (X \times CY)$. If a Hopf space G acts on X and Y it acts diagonally on $X * Y$ and hence on the n-**fold join** X^{*n} defined inductively by $X^{*0} = X$ and $X^{*(n+1)} = (X^{*n}) * X$. The purpose of this section is to prove

THEOREM 3.1: *Let F be the homotopy fibre of a map f: $X \to Y$ between connected CW complexes. If* $\operatorname{cat} f \le m$ *then there is an* ΩX*-equivariant map,*

$$F \to (\Omega X)^{*m}.$$

Remarks 3.2: (i) As we shall see, this assertion is essentially implicit in two results of Ganea.

(ii) It is reasonably clear that the existence of the equivariant map $F \to (\Omega X)^{*m}$ *characterizes* maps of category $\le m$. We have not included a proof since we do not need the stronger result. ∎

Proof of 3.1: Recall that the **fibre-cofibre construction** of Ganea, applied to a fibration $E \xrightarrow{p} X$ over a pointed space (X, a) consists of extending p to

$$\psi = p \cup \text{const.map}\colon E \cup_F CF \to X, \qquad F = p^{-1}(a)$$

and then converting ψ to a fibration $p'\colon E' \to X$ (as described in §2).

Iterating this construction (starting with the path space fibration for path connected (X, a)) Ganea produces the sequence of fibrations

$$\begin{array}{ccccccccc}
\Omega X & & F_1 & & F_2 & & & & F_m \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & & & \downarrow{\scriptstyle i_m} \\
PX & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_m \\
& {\scriptstyle p_0}\searrow & & {\scriptstyle p_1}\searrow & \downarrow{\scriptstyle p_2} & \swarrow{\scriptstyle p_m} & & & \\
& & & & X & & & &
\end{array}$$

And in [11, Prop. 2.2] he shows that connected CW complexes X have category $\leq m$ if and only if the fibration p_m admits a cross-section. His argument, however, applies verbatim to maps $f\colon Y \to X$ and shows that $\operatorname{cat} f \leq m$ if and only if f factors as

$$f\colon Y \xrightarrow{g} E_m \xrightarrow{p_m} X$$

(provided Y is a path connected CW complex).

Thus in the situation of Theorem 3.1 we have g, and hence the commutative diagram

$$\begin{array}{ccccc}
Y \underset{X}{\times} MX & \xrightarrow{(g,\mathrm{id})} & E_m \underset{X}{\times} MX & \xrightarrow{\text{composition}} & E_m \\
& {\scriptstyle p}\searrow & & \swarrow{\scriptstyle p_m} & \\
& & X & &
\end{array}$$

in which p is the conversion of f to a fibration. In particular we have an ΩX-equivariant map

$$F \to F_m. \tag{3.3}$$

We now complete the proof by constructing an ΩX-equivariant weak homotopy equivalence

$$F_m \to (\Omega X)^{*m}. \tag{3.4}$$

To do so suppose $E' \xrightarrow{p'} X$ is obtained by converting some arbitrary map $Z \to X$ to a fibration, and apply the fibre-cofibre construction to obtain $E'' \xrightarrow{p''} X$. In [10; Theorem 1.1] Ganea shows that the fibre F'' of p'' has the weak homotopy type of $F' * \Omega X$, F' the fibre of p'. We need this equivariantly; explicitly

LEMMA 3.5: *There is an ΩX-equivariant weak homotopy equivalence*

$$F'' \to F' * \Omega X.$$

Proof of 3.5: Write

$$F'' = (E' \cup_{F'} CF') \times_X PX = (E' \times_X PX) \cup_{F' \times \Omega X} (CF' \times \Omega X).$$

This identifies F'' as the **double mapping cylinder** $T(i, \mathrm{pr}_2)$ of the inclusion i: $F' \times \Omega X \to E' \times_X PX$ and the second factor projection pr_2: $F' \times \Omega X \to \Omega X$.

On the other hand, composition gives weak homotopy equivalence

$$\phi\colon E' \times_X PX = Z \times_X MX \times_X PX \to Z \times_X PX = F',$$

and $\phi \circ i$ is the action a: $F' \times \Omega X \to F'$. Thus ϕ extends in the obvious way to a weak homotopy equivalence Φ: $F'' = T(i, \mathrm{pr}_2) \to T(a, \mathrm{pr}_2)$.

Moreover, a weak homotopy equivalence ψ: $F' \times \Omega X \to F' \times \Omega X$ is given by ψ: $(v, \omega) \mapsto (v \cdot \omega, \omega)$. It satisfies $\mathrm{pr}_1 \circ \psi = a$ and $\mathrm{pr}_2 \circ \psi = \mathrm{pr}_2$, and so extends naturally to a weak homotopy equivalence Ψ: $T(a, \mathrm{pr}_2) \to T(\mathrm{pr}_1, \mathrm{pr}_2)$.

But $T(\mathrm{pr}_1, \mathrm{pr}_2)$ is precisely $F' * \Omega X$, and it is straightforward to verify that $\Psi \circ \Phi$ is ΩX-equivariant with respect to the diagonal action on $F' * \Omega X$. ∎

We now revert to the proof of Theorem 3.1. An obvious induction using Lemma 3.5 gives (3.4), and (3.3) and (3.4) give the Theorem. ∎

4. Chain factorization

Our goal is to translate Theorem 3.1 into a result on $C_*(\Omega X)$-modules. This requires two notions of "resolution", one for DGA modules (due essentially to Moore [15]) and one for DGH-modules (which goes back to the original constructions of group cohomology).

Suppose then that M is a right A-module, A a DGA. Denote by wM the R-free chain complex on the basis $\{e_z, e'_x, e''_x | z \text{ a cycle in } M, x \in M\}$ with $de_z = 0$, $de'_x = e''_x$ ($|e_z| = |z|$, $|e'_x| = |x|$). Define A-modules W_nM and A-linear maps

$$(4.1) \qquad \xrightarrow{\partial_{n+2}} W_{n+1}M \xrightarrow{\partial_{n+1}} W_nM \to \cdots \xrightarrow{\partial_1} W_oM \xrightarrow{\varepsilon_M} M \to 0$$

by:

(i) $W_oM = wM \otimes A$, $\varepsilon_M\colon e_z, e'_x, e''_x \mapsto z, x, dx$,

(ii) $W_1M = sW_o(\ker \varepsilon_M)$, $\partial_1(su) = \varepsilon_{\ker \varepsilon_M}(u)$, and

(iii) $W_{n+1}M = sW_o(\ker \partial_n)$; $\partial_{n+1}(su) = \varepsilon_{\ker \partial_n}(u)$.

Passage to homology gives the sequence

(4.2)

$$\to H(W_{n+1}M) \xrightarrow{H(\partial)} H(W_nM) \to \cdots \to H(W_oM) \xrightarrow{H(\varepsilon_M)} H(M) \to 0.$$

Denote the differential in each W_nM by ∂_I, so that

$$WM = (\bigoplus_{n \geq 0} W_nM, d = \partial_I + \partial)$$

is an A-module, filtered by the submodules $W_{\leq n}M$. Extend ε_M (by zero in W_+M) to a morphism $\varepsilon_M\colon WM \to M$ of A-modules. We shall need the

LEMMA 4.3: *Let A be a DGA and M be an A-module. Then*

(i) *Each (W_nM, ∂_I) is A-free.*

(ii) *The sequence (4.1) is exact and $\varepsilon_M\colon WM \to M$ is a homology isomorphism.*

(iii) *The sequence (4.2) is an $H(A)$-free resolution of $H(M)$.*

(iv) *The functor $\mathrm{Hom}_A(WM; -)$ preserves exact sequences and homology isomorphisms.*

Proof: The first three assertions are immediate and the exactness of

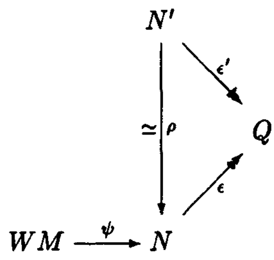
$$\mathrm{Hom}_A(WM; -) = \prod_n \mathrm{Hom}_A(W_nM; -)$$

follows from the fact that each W_nM is A-free. Now every A-module morphism is the composite of an injective homology isomorphism and a surjection (both of A-modules) and so we need only show $\mathrm{Hom}_A(WM; \phi)$ is a homology isomorphism whenever ϕ is an injective or surjective homology isomorphism.

Since $\mathrm{Hom}_A(WM; -)$ is exact it suffices to show that $H(\mathrm{Hom}_A(WM; N)) = 0$ if $H(N) = 0$. Suppose α: $WM \to N$ is a cycle and $\alpha|_{W_{<n}M} = d\theta$. By construction $W_nM = (Z \oplus C) \otimes A$ with Z, C both R-free, d: $Z \to W_{<n}M$ and d: $C \to (Z \otimes A) \oplus W_{<n}M$.

Recall now that $d\theta = d \circ \theta + (-1)^{|\alpha|}\theta \circ d$. We observe that $\alpha - (-1)^{|\alpha|}\theta \circ d$: $Z \to$ cycles of N. Since $H(N) = 0$ and Z is R-free we can extend θ to Z so that $\alpha = d\theta$ in Z. Extend θ to an A-linear map in $Z \otimes A$ and repeat the argument to extend it to $C \otimes A$; i.e., to all of $W_{\leq n}$. Conclude by induction that $\alpha = d\theta$. ∎

Remark 4.4: If we are given the commutative diagram of A-modules



in which $H(\rho)$ is an isomorphism and ε and ε' are surjective then we can construct $\phi\colon WM \to N'$ such that $\rho\phi$ is A-homotopic to ψ and $\varepsilon' \circ \phi = \varepsilon \circ \psi$.

In fact, Lemma 4.3(iv) gives $\phi'\colon WM \to N'$ and $\rho\phi' - \psi = d\theta$ with a morphism $\theta \in \mathrm{Hom}_A(WM; N)_1$. Because $\mathrm{Hom}_A(WM; -)$ is exact we can find

$$\theta' \in \mathrm{Hom}_A(WM; N')$$

such that $\varepsilon' \circ \theta' = \varepsilon \circ \theta$. Set $\phi = \phi' - d\theta'$. ∎

Next, suppose A is a differential graded Hopf algebra with augmentation ideal I. Define right A-modules R_nA by $R_nA = A \otimes (sI)^{\otimes n}$, $n \geq 0$ (tensor product of chain complexes with diagonal action of A), and denote the differentials by ∂_I. Again we have a sequence of A-linear maps

$$\to R_{n+1}A \xrightarrow{\partial} R_nA \to \cdots \to R_0A = A \xrightarrow{\varepsilon} R \to 0 \tag{4.5}$$

in which $\partial(x \otimes sy_1 \otimes \cdots \otimes sy_n) = \varepsilon(x)y_1 \otimes sy_2 \otimes \cdots \otimes sy_n$. Again $RA = (\bigoplus_{n\geq 0} R_nA, \partial = \partial_I + \partial)$ is an A-module filtered by the submodules $R_{\leq n}A$ and again ε extends (by zero in R_+A) to a morphism $\varepsilon\colon RA \to R$.

LEMMA 4.6: *The sequence (4.5) is exact and $\varepsilon\colon RA \to R$ is a chain equivalence.*

Proof: Define $h(x \otimes sy_1 \otimes \cdots \otimes sy_n) = 1 \otimes sx \otimes sy_1 \otimes \cdots \otimes sy_n$; then $h\partial_I + \partial_I h = 0$ and $h\partial + \partial h = id - \eta\varepsilon$. ∎

Note next that for any right A-module M we can extend ε to the morphism

$$\mathrm{id}_M \otimes \varepsilon\colon M \otimes RA \to M \tag{4.7}$$

of A-modules, with A acting diagonally in $M \otimes RA$. We can now state the main result of this section.

THEOREM 4.8: *Let F be the homotopy fibre of a map $f\colon X \to Y$ between connected CW complexes. If $\operatorname{cat} f \le m$, then there is a commutative diagram of $C_*(\Omega X)$-modules:*

$$\begin{array}{ccc} WC_*(F) & \xrightarrow{\ \phi\ } & C_*(F)\otimes R_{\le m}C_*(\Omega X) \\ & {\scriptstyle \varepsilon}\searrow \qquad \swarrow{\scriptstyle \mathrm{id}\otimes\varepsilon} & \\ & C_*(F) & \end{array}$$

Remark 4.9: Suppose $f = \mathrm{id}_X$ and X is a simply connected rational space with finite (rational) betti numbers. The existence of a diagram as in Theorem 4.8 (here $F = PX$!) implies $M \operatorname{cat} X \le m$ and hence by a theorem of Hess [12] together with [4; Theorem VIII] $\operatorname{cat} X \le m$. Thus in this case the existence of the diagram is equivalent to $\operatorname{cat} X \le m$. A recent example of Idrissi [13] shows, however, that this equivalence fails for maps $f\colon Y \to X$, even in the simply connected rational case. ∎

Proof of 4.8: Our first step is to show that $R_{\le m}C_*(\Omega X)$ is connected to $C_*((\Omega X)^{*m})$ by a sequence of $C_*(\Omega X)$-linear chain equivalences of augmented modules. In fact, given Proposition 2.9 (ii) it is sufficient to observe that a $C_*(\Omega X)$-linear chain equivalence of augmented modules,

$$R_{\le k}C_*(\Omega X) * C_*(\Omega X) \xrightarrow{\ \simeq\ } R_{\le k+1}C_*(\Omega X)$$

is given by ($C = C_*(\Omega X)$, I=augmentation ideal):

$$\begin{aligned} (R_{\le k}C) * C &= [c(R_{\le k}C)\otimes C] \oplus [R_{\le k}C \otimes sI] \qquad \text{(as R-modules)} \\ &\to C \oplus [R_{\le k}C \otimes sI] \\ &= R_{\le k+1}C. \end{aligned}$$

On the other hand, Theorem 3.1 gives a $C_*(\Omega X)$-linear map $\tau\colon C_*(F) \to C_*((\Omega X)^{*m})$ and Proposition 2.9 (i) asserts that $\Delta\colon C_*(F) \to C_*(F)\otimes C_*(F)$ is also $C_*(\Omega X)$-linear. We assemble all this in the diagram of $C_*(\Omega X)$-modules:

$$\begin{array}{ccccccccc} & & & & & & C_*(F)\otimes R_{\le m}C_*(\Omega X) & & \\ & & & & & & \uparrow{\scriptstyle\simeq} & \searrow & \\ & & & & & & \vdots & \longrightarrow & C_*(F) \\ & & & & & & \downarrow{\scriptstyle\simeq} & \nearrow{\scriptstyle \mathrm{id}\otimes\varepsilon} & \\ WC_*(F) & \xrightarrow{\ \varepsilon\ } & C_*(F) & \xrightarrow{\ \Delta\ } & C_*(F)\otimes C_*(F) & \xrightarrow{\ \mathrm{id}\otimes\tau\ } & C_*(F)\otimes C_*((\Omega X)^{*m}) & & \end{array}$$

Evidently the composite $C_*(F) \to C_*(F) \otimes C_*(F) \to C_*(F) \otimes C_*((\Omega X)^{*m}) \to C_*(F)$ is just $\mathrm{id}_{C_*(F)}$, and so we can apply Remark 4.4 to achieve the proof of the theorem. ∎

5. Homological calculations

Here we complete the proof of Theorem A and its corollary, with the aid of Theorem 4.8 and the homological calculations of the title. These take place in the following context: we fix a differential graded Hopf algebra A and a right A-module M, subject to the following restrictions:

(5.1) $H(M)$ and $H(A)$ are R-free, and concentrated in non-negative degrees.

(5.2) $H(A)$ is a Hopf algebra with conjugation.

(5.3) There is a commutative diagram of A-modules,

$$\begin{array}{ccc} WM & \xrightarrow{\ \phi\ } & M \otimes R_{\leq m} A \\ & {\scriptstyle \varepsilon_M} \searrow \quad \swarrow {\scriptstyle \mathrm{id} \otimes \varepsilon} & \\ & M & \end{array}$$

Then we have

THEOREM 5.4: *Under the hypotheses (5.1), (5.2) and (5.3),*

$$\text{pos.proj.grade}_{H(A)}(H(M)) \leq m.$$

If equality holds then

$$\text{pos.proj.grade}_{H(A)}(H(M)) = \text{proj.dim}_{H(A)}(H(M)).$$

Before proving the theorem we note that if $f\colon Y \to X$ is a continuous map of connected CW complexes with homotopy fibre F, and if $H(F)$ and $H(\Omega X)$ are R-free, then (5.1) and (5.2) are satisfied by $C_*(F)$ and $C_*(\Omega X)$. If, in addition, $\operatorname{cat} f \leq m$ then (5.3) is exactly Theorem 4.8. Thus Theorem A follows from Theorem 5.4.

Proof of 5.4: Filter WM and $M \otimes R_{\leq m} A$ respectively by the submodules $W_{\leq n} M$ and $M \otimes R_{\leq n} A$. Then, although ϕ (in (5.3)) need not preserve filtrations, it raises them by at most m. The proof of the theorem consists essentially of the next two lemmas.

LEMMA 5.5: *Suppose* pos.proj.grade$_{H(A)}(H(M)) \geq m$. *Then the map* ϕ *in (5.3) can be chosen to preserve filtrations.*

Proof 5.5: Suppose ϕ raises filtration degree by at most k $(1 \leq k \leq m)$. Then it induces a "morphism" $E^i(\phi)$: $E^i_{p,q}(WM) \to E^i_{p+k,q-k}(M \otimes R_{\leq m})$ of spectral sequences. For simplicity we put $E^i(M \otimes R_{\leq m} A) = E^i$.

Now observe that we can regard $E^i(\phi) \in \mathrm{Hom}_{H(A)}(E^i(WM); E^i)_{k,-k}$ as a cycle of bidegree $(k, -k)$ in the chain complex of bigraded $H(A)$-linear maps. Moreover, since

$$E^1_{q,*} = \begin{cases} H(M) \otimes H(A) \otimes H(sI)^{\otimes q} & q \leq m \\ 0 & q > m \end{cases}$$

there are short exact sequences of chain complexes

$$0 \to E^1_{<q,*} \to E^1_{\leq q,*} \to (H(M) \otimes H(A) \otimes H(sI)^{\otimes q}) \to 0, \qquad q \leq m.$$

And since each $E^1_{p,*}(WM) = H(W_pM)$ is $H(A)$-free, this sequence is transformed by $\mathrm{Hom}_{H(A)}(E^1(WM); -)_{*,*}$ into a short exact sequence of chain complexes terminating in $C(q) = \mathrm{Hom}_{H(A)}(E^1(WM);\ H(M) \otimes H(A) \otimes H(sI)^{\otimes q})_{*,*}$. But $(E^1(WM), d^1)$ is the $H(A)$-free resolution (4.2) of $H(M)$. Thus

$$H_{s,t}(C(q)) = \mathrm{Ext}_{H(A)}^{-s+q,-q-t}(H(M);\ H(M) \otimes H(A) \otimes H(sI)^{\otimes q}).$$

Finally, since $H(A)$ has a conjugation by (5.2), Lemma 2.1 asserts that $H(M) \otimes H(A) \otimes H(sI)^{\otimes q}$ (diagonal action) is a free $H(A)$-module on a basis of elements of degree ≥ 0. Thus the hypothesis pos.proj.grade$_{H(A)}(H(M)) \geq m$ gives $H_{s,t}(C(q)) = 0$, $s > 0$. Since this is true for each $q \leq m$ it follows that $H_{s,*}(\mathrm{Hom}_{H(A)}(E^1(WM); E^1)) = 0$, $s > 0$. In particular, $E^1(\phi) = d\alpha$ for some $\alpha \in \mathrm{Hom}_{H(A)}(E^1(WM); E^1)_{k+1,-k}$.

Next, identify

$$(E^0(WM), d^0) = (WM, \partial_I) \text{ and } (E^0, d^0) = (M \otimes R_{\leq m} A,\ d \otimes 1 \pm 1 \otimes \partial_I).$$

Thus $E^0_{p,*}(WM) = W_pM$ has the form $L(p) \otimes A$ for some R-free chain complex $L(p)$ with R-free homology. It follows that the restriction of α to each $H(L(p))$ has the form $H(\sigma_p)$ for some d^0-cycle $\sigma_p \in \mathrm{Hom}(L(p); E^0_{p+k+1,*})$. Extend σ_p by A-linearity to all of W_pM and put $\sigma = \{\sigma_p\}$ and $\phi' = \phi - d\sigma$. Then $(\mathrm{id} \otimes \varepsilon)\phi' = \varepsilon_M$ and $E^1(\phi') = 0$.

But since $E^1(\phi') = 0, E^0(\phi')$ restricted to $L(p)$ has the form $d^o h_p$ for some $h_p\colon L(p) \to E^0_{p+k,*}$. Put $\phi'' = \phi' - dh$, verify again that $(\mathrm{id} \otimes \varepsilon)\phi'' = \varepsilon_M$ and notice that ϕ'' raises filtration degree by at most $k - 1$. ∎

LEMMA 5.6: *If the map ϕ in (5.3) preserves filtrations then*

$$\mathrm{proj.dim}_{H(A)}(H(M)) \leq m.$$

Proof: Consider the same spectral sequences as in Lemma 5.6 and note that we have

$$\begin{array}{ccccc} H_{**}(WM, \partial_I) & \xrightarrow{E^1(\phi)} & H_{**}(M \otimes R_{\leq m}A, \delta) & \xrightarrow{\text{inclusion}} & H_{**}(M \otimes RA, \delta) \\ & {\scriptstyle H(\varepsilon_M)} \searrow & & \swarrow {\scriptstyle \mathrm{id} \otimes H(\varepsilon)} & \\ & & H(M) & & \end{array}$$

where $\delta = d \otimes 1 \pm 1 \otimes \partial_I$. It is straightforward to verify that

$$H(\varepsilon)\colon (H(RA, \partial_I), \partial) \to R$$

is a chain equivalence, and so by Lemma 2.1 $\mathrm{id} \otimes H(\varepsilon)$ is an $H(A)$-free resolution of $H(M)$. Thus the diagram above shows that the lift of $\mathrm{id}_{H(M)}$ to a map of $H(A)$-free resolutions factors through a complex of length m. Hence $\mathrm{proj.dim}_{H(A)}(H(M)) \leq m$. ∎

To complete the proof of Theorem 5.4 it suffices to recall (1.1) that

$$\mathrm{pos.proj.grade}_{H(A)} \leq \mathrm{proj.dim}_{H(A)}(H(M))$$

and to apply the two lemmas. ∎

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